Recursive Contracting in Labor

1 Contracting with shocks

1.1 Overview of the environment

We consider a mass of workers with productivities y_{it} that evolves over time according to a Markov process. Workers are risk averse with utility $u(\cdot)$. There is also a measure of firms who compete for workers. Each firm can hire exactly one worker.

1.2 Contracting with shocks: some notations

We define the history of shocks to the worker $y^t = (y_0^t....y_t^t)$, $y^t \in \mathcal{H}_t \subset \mathbb{R}^{t+1}$. In addition we have a probability associated with each history that we write $\pi_t(y^t)$ and where we have that $\int_{y^t} \pi^t(y^t)dy^t = 1$. We also define $\mathcal{H}_t(y^\tau) \subset \mathcal{H}_t$ the set of histories that start with y^τ:

$$\mathcal{H}_t(y^\tau) \equiv \{y^t \in \mathcal{H}_t \text{ s.t. } y_i^t{=}y_i^\tau \text{ for } i \leq \tau\}$$

and for any $y^t \in \mathcal{H}_t(y^\tau)$ we define $\pi_t(y^t|y^\tau) = \pi_t(y^t)/\pi_t(y^\tau)$. A contract $\mathcal{C}$ defines consumption after each history. More precisely:

$$\mathcal{C} = \{w_t(\cdot) \text{ for } t \geq 0\} \text{ where } w_t : \mathcal{H}_t \to \mathbb{R}$$

w_t is a function that takes $t+1$ arguments. We will also have to refer to the contract starting at a sub history, consider $\tilde{\mathcal{C}} = \{\tilde{w}_t(\cdot) \text{ for } t \geq 0\}$

$$\tilde{\mathcal{C}} = \mathcal{C}(y^t) \text{ iif } \forall \tau \geq 0,\ \forall y^{t+\tau} \in \mathcal{H}_{t+\tau}(y^t),\ \tilde{w}_\tau(y_t^{t+\tau}, ..., y_{t+\tau}^{t+\tau}) = w_{t+\tau}(y^{t+\tau})$$

We define the value of a contract to the firm $P(\mathcal{C})$ and to the agent $V(\mathcal{C})$.

We have that

$$\begin{aligned} P(\mathcal{C}, y_0) &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(y_t^t - w_t(y^t) \right) \\ &= \sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t (y_t^t - w_t(y^t)) \pi_t(y^t|y_0) \end{aligned}$$

and the value to the worker is

$$\begin{aligned} V(\mathcal{C}, y_0) &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c^t) \\ &= \sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t u(w_t(y^t))) \pi_t(y^t|y_0) \end{aligned}$$

1.3 Solution with full commitment

We can first look at the contract offered if both sides can commit, meaning that they can agree to a contract that will give values below what contacting a new firm would give. We look for the solution to the following problem:

$$P(y_0, v) = \max_{\mathcal{C}} P(\mathcal{C}, y_0)$$
$$\text{s.t.} V(\mathcal{C}, y_0) \geq v$$

which gives the following Lagrangian:

$$\sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t (y_t^t - w_t(y^t)) \pi_t(y^t|y_0) - \lambda \left(\sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t u(w_t(y^t))) \pi_t(y^t|y_0) - v \right)$$

where the FOC for $w_t(y^t)$ gives

$$u'(w_t(y^t)) = \frac{1}{\lambda}$$

which delivers a fixed wage, we can then solve for this $\bar{w}$

$$\sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t u(\bar{w}) \pi_t(y^t|y_0) = v$$
$$u(\bar{w}) = (1 - \beta) v$$

Finally we can impose a zero profit condition:

$$P(y_0, v) = \sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t (y_t^t - \bar{w}) \pi_t(y^t|y_0)$$
$$= \bar{y}(y_0) - \frac{u^{-1}\left((1-\beta)v\right)}{1-\beta} = 0$$

where $\bar{y}(y_0)$ is the present discounted value of output conditional on starting at y_0. We then get that the equilibrium wage is given by

$$\bar{w}(y_0) = u^{-1}(u((1-\beta)\bar{y}(y_0)))$$

1.4 One sided limited commitment

We consider the following problem:

$$P(y_0, v) = \sup_{\mathcal{C}} \mathbb{E}_0 \sum_{t=0}^{\infty} y_t^t - w_t(y^t)$$
$$\text{s.t.} V(\mathcal{C}, y_0) \geq v$$
$$\forall t \geq 1, y^t \quad V(\mathcal{C}(y^t), y_t^t) \geq \underline{v}(y_t^t)$$

Definition 1. The equilibrium is a set of contracts $\mathcal{C}(y)$ and values $\underline{\mathrm{v}}(y)$ such that:

1. $\mathcal{C}(y)$ solves the firm problem expressed above
2. $\underline{\mathrm{v}}(y)$ sets firms profit to 0: $P(y, \underline{\mathrm{v}}(y)) = 0$

To solve the problem, we associate multipliers $\mu(y_0)$ and $\beta^t \lambda^t(y^t)\pi_t(y^t|y_0)$ with each of these constraints, then we can write the Lagrangian

$$
\begin{aligned}
L(y_0, v, \mathcal{C}, \mu, \{\lambda^t\}_{t=1..\infty}) = &\sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t (y_t^t - w_t(s^t))\pi_t(y^t|y_0) \\
&- \mu(y_0)\left(v - V(\mathcal{C}, y_0)\right) \\
&- \sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t \lambda^t(y^t) \left(\underline{\mathrm{v}}(y_t^t) - V(\mathcal{C}(y^t), y_t^t)\right) \pi_t(y^t|y_0)
\end{aligned}
$$

1.4.1 Finding a recursive formulation to the problem

we then split the firm sum to extract $t = 0$ terms. We then split the λ^t in period 1 and the rest and get:

$$
\begin{aligned}
L = &\, y_0 - w_0(y_0) - \mu\left(v - V(\mathcal{C}, y_0)\right) \\
&+ \sum_{t=1}^{\infty} \sum_{y^1} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t (y_t^t - w_t(y^t))\pi_t(y^t|y_0)\pi_t(y^t|y_0) \\
&- \sum_{t=1}^{\infty} \sum_{y^1} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t \lambda^t(y^t) \left(\underline{\mathrm{v}}(y_t^t) - V(\mathcal{C}(y^t), y_t^t)\right) \pi_t(y^t|y_0)
\end{aligned}
$$

we then define then introduce the following variables:

$$
\forall y^1 \in \mathcal{H}_1(y_0),\ v(y^1) \equiv V(\mathcal{C}(y^1), y_1^1)
$$

and note that we have that

$$
\begin{aligned}
V(\mathcal{C}, y_0) &= \sum_{t=0}^{\infty} \sum_{y^t \in \mathcal{H}_t(y_0)} \beta^t u(w_t(s^t))) \pi_t(y^t|y_0) \\
&= u(w_0(y_0)) + \sum_{t=1}^{\infty} \sum_{y^1 \in \mathcal{H}_1(y_0)} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t u(w_t(s^t))) \pi_t(y^t|y_0) \\
&= u(w_0(y_0)) + \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} \left(\sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^{t-1} u(w_t(s^t))) \pi_t(y^t|y^1) \right) \pi_t(y^1|y_0) \\
&= u(w_0(y_0)) + \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} V(\mathcal{C}(y^1), y_1^1) \pi_1(y^1|y_0) \\
&= u(w_0(y_0)) + \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} v(y^1) \pi_1(y^1|y_0) \\
&= u(w_0(y_0)) + \beta \sum_{y'} v(y') \pi(y^{'}|y_0)
\end{aligned}
$$

that we maximize over, so we can write

$$
\begin{aligned}
L \quad = \quad & y_0 - c_0 - \mu \left(v - u(w_0(y_0)) - \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} v(y^1) \pi_t(y^1|y_0) \right) \\
& - \sum_{y^1} \beta \lambda^1(y^1) \left(\underline{\mathrm{v}}(y_1^1) - V(\mathcal{C}(y^1), y_1^1) \right) \pi_1(y^1|y_0) \\
& + \sum_{t=1}^{\infty} \sum_{y^1} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t (y_t^t - w_t(y^t)) \pi_t(y^t|y_0) \pi_t(y^t|y_0) \\
& - \sum_{y^1} \sum_{t=2}^{\infty} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t \lambda^t(y^t) \left(\underline{\mathrm{v}}(y_t^t) - V(\mathcal{C}(y^t), y_t^t) \right) \pi_t(y^t|y_0) \\
& - \sum_{y^1 \in \mathcal{H}_1(y_0)} \beta \mu(y^1) \left(v(y^1) - V(\mathcal{C}(y^1), y_1^1) \right) \pi_1(y^1|y_0)
\end{aligned}
$$

or

$$
\begin{aligned}
L &= y_0 - c_0 - \mu\left(v - u(w_0(y_0)) - \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} v(y^1)\pi_t(y^1|y_0)\right) \\
&\quad - \sum_{y^1} \beta\lambda^1(y^1)\left(\underline{\mathrm{v}}(y_1^1) - V(\mathcal{C}(y^1), y_1^1)\right)\pi_1(y^1|y_0) \\
&\quad + \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} \Big[\sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t (y_t^t - w_t(y^t))\pi_t(y^t|y_0)\pi_t(y^t|y^1) \\
&\quad - \sum_{t=2}^{\infty} \sum_{y^t \in \mathcal{H}_t(y^1)} \beta^t\lambda^t(y^t)\left(\underline{\mathrm{v}}(y_t^t) - V(\mathcal{C}(y^t), y_t^t)\right)\pi_t(y^t|y^1) \\
&\quad - \mu(y^1)\left(v(y^1) - V(\mathcal{C}(y^1), y_1^1)\right)\Big]\pi_1(y^1|y_0)
\end{aligned}
$$

or further:

$$
\begin{aligned}
L(y_0, v, \mathcal{C}, \mu, \{\lambda^t\}_{t=1..\infty}) &= y_0 - c_0 - \mu\left(v - u(w_0(y_0)) - \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} v(y^1)\pi_t(y^1|y_0)\right) \\
&\quad - \sum_{y^1} \beta\lambda^1(y^1)\left(\underline{\mathrm{v}}(y_1^1) - V(\mathcal{C}(y^1), y_1^1)\right)\pi_1(y^1|y_0) \\
&\quad + \beta \sum_{y^1 \in \mathcal{H}_1(y_0)} L(y_1^1, v(y^1), \mathcal{C}(y^1), \mu(y_1), \{\lambda^t\}_{t=2..\infty})\pi_1(y^1|y_0)
\end{aligned}
$$

in other words we can write our problem recursively as:

$$
\begin{aligned}
P(y, v) \quad &= \sup_{w, v(y')} \quad y - w + \beta \mathbb{E} P(y, v(y')) \\
&\quad s.t. \qquad v = u(w) - \beta \mathbb{E} v(y') \\
&\qquad\qquad\quad v(y') \geq \underline{\mathrm{v}}(y') \quad \forall y'
\end{aligned}
$$

solve this problem and get the Harris and Holmstrom (1982) solution. Using promised utility was introduced generally by Spear and Srivastava (1987).

1.4.2 The shape of the contract

We take the first order conditions to the recursive problem:

$$
\begin{aligned}
L = y - w + \beta \sum_{y'} P(y', v(y'))\pi(y'|y) \\
- \lambda\left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) - v\right) \\
- \beta \sum_{y_1} \phi(y')\left(v(y') - \underline{\mathrm{v}}(y')\right)\pi(y'|y)
\end{aligned}
$$

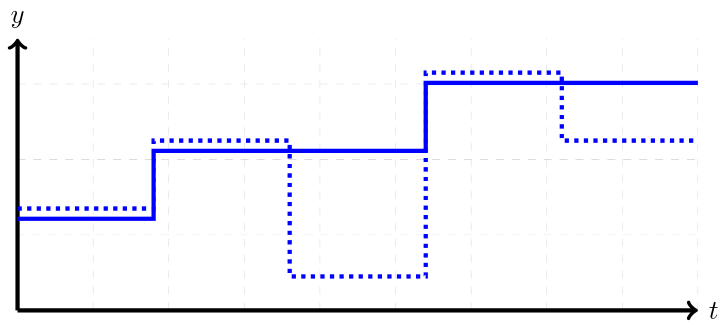

Figure 1: wage path with one-sided limited commitment

we get:

$$u'(w) = \frac{1}{\lambda}$$
$$P_v(y', v(y')) + \lambda - \phi(y') = 0$$

together with the envelope condition

$$P_v(y, v) = \lambda$$

and so we can write that

$$\frac{1}{u'(w(y'))} - \frac{1}{u'(w)} = \phi(y')$$

We then go a step further and use the complementary slackness conditions that says that $\phi(y') = 0$ or $v(y') - \underline{v}(y') \geq 0$. When the constraint is not binding we get that $w' = w$ and that $v(y')$ solves $P_v(y', v(y')) = P_v(y, v)$. In the case where the constraint binds, then we know that $v(y') = \underline{v}(y')$. So define the following:

$$w^*(y) \text{ s.t. } \frac{1}{u'(w^*(y))} = P_v(y, \underline{v}(y))$$

then we get that:

$$w(y') = \max\{w, w^*(y')\}$$

1.5 2 sided limited commitment

We modify the problem to have two sided limited commitment as well as mobility costs. We end up with the following contracting problem:

$$\begin{aligned} P(y,v) \quad &= \sup_{w,v(y')} \quad y - w + \beta \mathbb{E} P(y, v(y')) \\ &s.t. \quad v = u(w) - \beta \mathbb{E} v(y') \\ & \qquad v(y') \geq \underline{v}(y') \quad \forall y' \\ & \qquad P(y', v') \geq 0 \end{aligned}$$

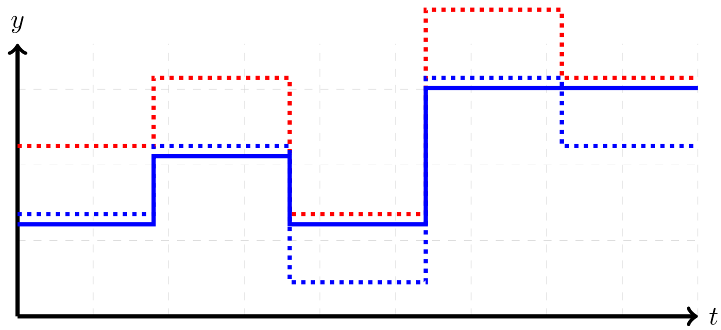

Figure 2: wage path with two-sided limited commitment

and the free entry condition is modified to

$$P(y, \underline{v}(y)) = c$$
$$P_v(y, \underline{v}(y)) = \frac{1}{u'(\underline{w}(y'))}$$

Working out the first order conditions in these cases will deliver a very close solution, however there will also be an upper bound to the wage:

$$w(y') = \min\left\{\max\left\{w, \underline{w}(y')\right\}, \overline{w}(y')\right\}$$

where $\overline{w}(y')$ is the wage associated with the firm firm making zero profits:

$$P(y, \overline{v}(y)) = 0$$
$$P_v(y, \overline{v}(y)) = \frac{1}{u'(\overline{w}(y'))}$$

1.5.1 Using recursive Lagrangian formulation

We notice that the problem seems to be more natural when expressed in marginal utilities. This approach has been developed by Marcet and Marimon (2011). To do so we consider a transformation of the problem as follows:

$$\mathcal{L}(y, \rho) = \max_v P(y, v) + \rho v.$$

We then develop it forward:

$$\mathcal{L}(y, \rho) = \max_v P(y, v) + \rho v.$$

$$
\begin{aligned}
\mathcal{L}(y,\rho) \quad &= \max_v \sup_{w,v(y')} \quad y - w + \beta \sum_{y'} P(y, v(y'))\pi(y'|y) + \rho v \\
&s.t. \qquad\qquad v = u(w) + \beta \sum_{y'} v(y')\pi(y'|y) \\
&\qquad\qquad\qquad v(y') \geq \underline{\mathrm{v}}(y') \quad \forall y'
\end{aligned}
$$

We can write the full Lagrangian:

$$
\begin{aligned}
L = y - w + \beta \sum_{y'} P(y', v(y'))\pi(y'|y) + \rho v \\
- \lambda \left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) - v \right) \\
- \beta \sum_{y_1} \phi(y') \left(v(y') - \underline{\mathrm{v}}(y') \right) \pi(y'|y)
\end{aligned}
$$

$$
\begin{aligned}
L = y - w + \beta \sum_{y'} P(y', v(y'))\pi(y'|y) + \rho \left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) \right) \\
- \lambda \left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) - v \right) \\
- \beta \sum_{y_1} \phi(y') \left(v(y') - \underline{\mathrm{v}}(y') \right) \pi(y'|y)
\end{aligned}
$$

$$
\begin{aligned}
L = y - w + \beta \sum_{y'} P(y', v(y'))\pi(y'|y) + \rho \left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) \right) \\
- \lambda \left(u(w) - \beta \sum_{y'} v(y')\pi(y'|y) - v \right) \\
- \beta \sum_{y_1} \phi(y') \left(v(y') - \underline{\mathrm{v}}(y') \right) \pi(y'|y)
\end{aligned}
$$

1.6 Back loading

We consider a slightly different environment. First we remove the sources of shocks. Second we are going to constrain the meeting with a matching technology. Firms will post vacancies at cost c and worker will be able to search every period. Firms will post contracts $\mathcal{C}$ that will only specify wage payments as a function of calendar time (since there are no shocks). Implicitly we also assume that outside offers are not contractable.

Definition 2. The equilibrium is given by a set of contracts $\mathcal{C}$ and associated tightness θ, both indexed by the value v they delivered to the worker. The conditions are such that:

- $\mathcal{C}$ solves is the optimal problem for the firm
- $p(v)$ is generated by the free entry condition
- $\theta(v)$ clears each of the search markets

The reader can look at Shi (2008) for the full exposition of this environment. We consider the search decision of the worker:

$$\max_{v_1} p(v_1)(v_1 - v')$$

which gives FOC:

$$p'(v_1)(v_1 - v') - p(v_1) = 0$$

the return to search is given by

$$\tilde{r}(v') = p(v_1)(v_1 - v') + v'$$

and its derivative is equal to

$$\tilde{r}_v(v') = 1 - p(v_1) = \tilde{p}(v')$$

$$\begin{aligned} P(v) \quad &= \sup_{v'} \quad y - w + \beta\tilde{p}(v')P(v') \\ &\quad s.t. \qquad v = u(w) + r(v') \end{aligned}$$

we then write the Lagrangian to get

$$y - w + \beta\tilde{p}(v')P(v') - \lambda\left(u(w) + \tilde{r}(v') - v\right)$$

and we derive the FOC to get:

$$\begin{aligned} u'(w) &= \frac{1}{\lambda} \\ \tilde{p}_v(v')P(v') + \tilde{p}(v')P_v(v') - \lambda r_v(v') &= 0 \end{aligned}$$

together with the envelope condition we can rewrite as:

$$\frac{\tilde{p}_v(v')}{\tilde{p}(v')}P(v') = \frac{1}{u'(w')} - \frac{1}{u'(w)}$$

we see that as long as the firm makes positive profit, the wage will increase over time, this referred to as backloading.

1.7 Re-introducing shocks and Recursive Lagrangian formulation

We re-introduce the shocks to the worker and seek a tractable solution. To that end we are going to re-fromulate the firm problem using promise marginal utility instead of promised utility. We have the following recursive formulation for $\mathcal{J}$

$$\begin{aligned} P(y,v) = & \sup_{W,W_{x'y'}} y - w + \beta\tilde{p}(v')\mathbb{E}P(y',v(y')) \\ & s.t \quad (\lambda) \quad 0 = u(w) + \tilde{r}(v') - v, \\ & \qquad (\gamma) \quad 0 = v' - \sum_{y'} v(y'). \end{aligned}$$

From which we can construct the Pareto problem

$$\mathcal{J}(y,\rho) = \sup_v P(y,v) + \rho v.$$

Formally, $\mathcal{P}$ is also the Legendre–Fenchel transform of $\mathcal{P}$. We seek a recursive formulation. I first substitute the definition of $\mathcal{J}$ and the constraint on λ in $\mathcal{P}$ to get

$$\begin{aligned} \mathcal{P}(y,\rho) = & \sup_{V,w,W,W_{y'}} f(y) - w + \beta\tilde{p}(W)\mathbb{E}\mathcal{J}(y',W_{y'}) + \rho V \\ & s.t \quad (\lambda) \quad 0 = u(w) + \tilde{r}(W) - V, \\ & \qquad (\gamma) \quad 0 = W - \mathbb{E}W_{y'}. \end{aligned}$$

at which point I can substitute in the V constraint:

$$\begin{aligned} \mathcal{P}(y,\rho) = & \sup_{V,w,W,W_{y'}} f(y) - w + \beta\tilde{p}(W)\mathbb{E}\mathcal{J}(y',W_{y'}) + \rho\left(u(w) + \tilde{r}(W)\right) \\ & s.t \quad (\gamma) \quad 0 = W - \mathbb{E}W_{y'}. \end{aligned}$$

then I append the constraint $(\gamma)$with weight $\beta\gamma\tilde{p}(x,W)$

$$\begin{aligned} \mathcal{P}(y,\rho) \quad = \quad & \inf_{\gamma} \sup_{V,w,W,W_{y'}} f(y) - w + \rho\left(u(w) + \tilde{r}(W)\right) \\ & -\gamma\beta\tilde{p}(W)(W - \mathbb{E}W_{y'}) \\ & +\beta\tilde{p}(W)\mathbb{E}\mathcal{J}(y',W_{y'}) \end{aligned}$$

which finally we recombine as

$$\begin{aligned} \mathcal{P}(y,\rho) \quad = \quad & \inf_{\gamma} \sup_{V,w,W,W_{y'}} f(y) - w + \rho\left(u(w) + \tilde{r}(W)\right) \\ & -\beta\gamma\tilde{p}(W) \\ & +\beta\tilde{p}(W)\mathbb{E}\mathcal{J}(y',W_{y'}) + \gamma\mathbb{E}W_{y'} \end{aligned}$$

the final step is to move the sup to the right hand side to get:

$$\begin{aligned}
\mathcal{P}(y,\rho) &= \inf_{\gamma} \sup_{w,W} f(y) - w + \rho\,(u(w) + \tilde{r}(W)) \\
&\quad -\beta\gamma\tilde{p}(W) \\
&\quad +\beta\tilde{p}(W)\mathbb{E}\left[\sup_{W_{y'}} \mathcal{J}(y', W_{y'}) + \gamma W_{y'}\right]
\end{aligned}$$

where we recognize the expression for $\mathcal{P}$ and so we are left with solving the following saddle point functional equation (SPFE):

$$\mathcal{P}(y,\rho) = \inf_{\gamma} \sup_{w,W} f(y) - w + \rho\,(u(w) + \tilde{r}(W)) \\ - \beta\gamma\tilde{p}(W) + \beta\tilde{p}(W)\mathbb{E}\mathcal{P}(y',\gamma). \quad \text{(SPFE)}$$

From the solution of this equation we can reconstruct the lifetime utility of the worker, and the profit function of the firm

$$\begin{aligned}
V(x,z,\rho) &= \frac{\partial \mathcal{P}}{\partial \rho}(c,z,\rho) \\
\mathcal{J}(x,z,v) &= \mathcal{P}(x,z,\rho^*(x,z,v)) - \rho^*(x,z,v)\cdot v.
\end{aligned}$$

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