Dynamic Programming

ECON 34430: Topics in Labor Markets

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Agenda

1. Introduction
   - Cake eating, direct approach and Bellman
   - Stochastic Dynamic programming
2. Numerical methods
   - Value function iteration
   - Policy function iteration
   - Continuous space and collocation
3. General formulation and properties
4. Estimation
   - Rust assumptions
   - CCP methods
   - General estimation

Introduction

Cake eating problem

- $t = 0$, have a cake of size W_0
- preferences are

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

- choose $(c_0 ... c_T)$ to maximize present value

$$\max_{c_0 ... c_T} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{s.t.} \sum_{t=0}^{T} c_t \leq W_0$$

Finite horizon - Direct approach

- write the Lagrangian

$$\sum_{t=0}^{T} \beta^t u(c_t) - \lambda\big(\sum_{t=0}^{T} c_t - W_0\big)$$

- take FOCs:

$$\forall t, \quad \beta^t u'(c_t) = \lambda \qquad \text{and} \qquad \sum_{t=0}^{T} c_t = W_0$$

- which implies the common Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})$$

- from which we get with log utility

$$c_t = \frac{\beta^t}{\lambda} = \beta^t W_0 \frac{1-\beta^{T+1}}{1-\beta}$$

- note that FOC is only necessary, need to impose to achieve optimality

Infinite horizon - Direct approach

- write the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t u(c_t) - \lambda\left(\sum_{t=0}^{\infty} c_t - W_0\right)$$

- take FOCs:

$$\forall t, \quad \beta^t u'(c_t) = \lambda \qquad \text{and} \qquad \sum_{t=0}^{\infty} c_t = W_0$$

- which still implies the common Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})$$

- from which we get with log utility

$$c_t = \frac{\beta^t}{\lambda} = \beta^t \frac{W_0}{1-\beta}$$

- note that FOC is only necessary, need to impose $\sum_{t=0}^{\infty} c_t = W_0$ to achieve optimality

Infinite horizon - Recursive formulation

- Define:

$$V(W_0) = \max_{c_0 \ldots c_\infty} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } \sum_{t=0}^{\infty} c_t \leq W_0$$

- next, we split the max into today and all of the future

$$V(W_0) = \max_{c_0} \left[u(c_0) + \beta \max_{c_1 \ldots c_\infty} \sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right]$$

$$\text{s.t. } \sum_{t=1}^{\infty} c_t \leq W_0 - c_0$$

- and so

$$V(W_0) = \max_{c} u(c) + \beta V(W_0 - c)$$

Infinite horizon - Recursive formulation 2

- We have:

$$V(W_0) = \max_c u(c) + \beta V(W_0 - c)$$

- where $V(\cdot)$ is not known, but defined implicitly.
- We need to solve for $c(W_0)$ and $V(W_0)$ for all W_0
- if W we can take FOC to get

$$u'(c) = \beta V'(W_0 - c)$$
$$V'(W_0) = \beta V'(W_0 - c)$$

- which gives us back our Euler equation $u'(c_t) = \beta u'(c_{t+1})$

Infinite horizon - Recursive formulation 3

- Is this sufficient for optimality of the original problem?
- Remember that the Euler equation alone was not suficient in the direct attack
- We can replace the consumption rule in to get

$$c_t = \beta^t c_0 = (1-\beta)\beta^t W_0$$

- In some cases, it would also require a **transversality condition**

Stochastic Dynamic programming

- We now want to consider the stochastic cace
- The agents will face a series of ϵ_t taste shocks, which follow a Markov Process $\pi_{ij} = Pr[\epsilon_t = j | \epsilon_{t-1} = i]$
- We denote a history of shocks as $h^t = (\epsilon_0 ... \epsilon_t)$ with $\epsilon_t(h^t) = \epsilon_t$ and we write $\pi(h^t)$ its overall probability
- Now for every history $\epsilon^t = (\epsilon_0 ... \epsilon_t)$ the agent can choose a consumption $c_t(\epsilon^t)$.
- We write the general problem as:

$$\max_{c_0..c_\infty} \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \epsilon_t(h^t) u(c_t(h^t)) \pi_t(h^t)$$

$$\text{s.t. } \forall t, h^t \qquad \sum_{\tau} c_\tau \left(h^t(0,\tau) \right) \leq W_0$$

Stochastic Dynamic programming

- We can write the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t \sum_{h^t} \epsilon_t(h^t) u(c_t(h^t)) \pi_t(h^t) - \sum_t \sum_{h^t} \lambda_{h^t} \left[\sum_\tau c_\tau \left(h^t(0,\tau) \right) - W_0 \right]$$

- and we can take FOC conditions for each $c_t(h^t)$

$$\beta^t \epsilon_t(h^t) u'(c_t(h^t)) \pi_t(h^t) - \sum_{\tau \geq t} \sum_{h^\tau \in \Gamma_\tau(h^t)} \lambda_{h^\tau}$$

where $\Gamma_\tau(h^t) = \{h^\tau \text{ s.t. } h^\tau(0,\tau) = h^t\}$

Stochastic Dynamic programming

- We can combine FOC of $c_t(h^t)$ and $c_{t+1}(h^{t+1})$

$$
\begin{aligned}
&\beta^t \epsilon_t(h^t) u'(c_t(h^t)) \pi_t(h^t) \\
&\qquad = \sum_{\tau \geq t} \sum_{h^\tau \in \Gamma_\tau(h^t)} \lambda_{h^\tau} \\
&\qquad = \sum_{\tau \geq t} \Big[\sum_j \lambda_{[h^t:\epsilon_j]} + \sum_{h^\tau \in \Gamma_\tau(h^t:\epsilon_j)} \lambda_{h^\tau} \Big] \\
&\qquad = \sum_j \sum_{\tau \geq t+1} \sum_{h^\tau \in \Gamma_\tau(h^t:\epsilon_j)} \lambda_{h^\tau} \\
&\qquad = \sum_j \beta^{t+1} \epsilon_{t+1}(h^t:\epsilon_j) u'(c_{t+1}(h^t:\epsilon_j)) \pi_t(h^t:\epsilon_j)
\end{aligned}
$$

Stochastic Dynamic programming

- we can rewrite

$$\epsilon_t u'(c_t(h^t)) = \beta \sum_j \epsilon_j u'(c_{t+1}(h^t{:}\epsilon_j)) \frac{\pi_t(h^t{:}\epsilon_j)}{\epsilon_t(h^t)\pi_t(h^t)}$$

$$\epsilon_t u'(c_t(h^t)) = \beta \sum_j \epsilon_j u'(c_{t+1}(h^t{:}\epsilon_j))\pi_{ij}$$

$$\epsilon_t u'(c_t) = \beta \mathbb{E}[\epsilon_j u'(c_{t+1})|\epsilon_t]$$

- where we recovered the Euler equation !

Stochastic Dynamic programing

- we go back to the original formulation

$$V(W_0, \epsilon_0) = \max_{c_0..c_\infty} \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \epsilon_t(h^t) u(c_t(h^t)) \pi_t(h^t)$$

$$\text{s.t. } \forall t, h^t \qquad \sum_{\tau} c_\tau \left(h^t(0,\tau)\right) \leq W_0$$

- we apply the same trick as before, we split the max into today and the future

$$\max_{c_0..c_\infty} \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \epsilon_t(h^t) u(c_t(h^t)) \pi_t(h^t) =$$

$$\max_{c_0} \epsilon_0 u(c_0) +$$

$$+\beta \max_{c_1 \ldots c_\infty} \sum_{t=0}^{\infty} \beta^t \sum_{h^{t+1} \in \Gamma_{t+1}(\epsilon_0)} \epsilon_{t+1}(h^{t+1}) u(c_{t+1}(h^{t+1})) \pi_{t+1}(h^{t+1})$$

Stochastic Dynamic programing

- and we split the second part as before according the first realization in the path

$$\sum_{t=0}^{\infty} \beta^t \sum_{h^{t+1} \in \Gamma_{t+1}(\epsilon_0)} \epsilon_t(h^{t+1}) u(c_{t+1}(h^{t+1})) \pi_{t+1}(h^{t+1})$$

$$= \sum_{t=0}^{\infty} \beta^t \sum_j \sum_{h^{t+1} \in \Gamma_{t+1}(\epsilon_0 : \epsilon_j)} \epsilon_{t+1}(h^{t+1}) u(c_{t+1}(h^{t+1})) \pi_t(h^{t+1})$$

$$= \sum_j \pi_{ij} \sum_{t=0}^{\infty} \beta^t \sum_{h^{t+1} \in \Gamma_{t+1}(\epsilon_0 : \epsilon_j)} \epsilon_{t+1}(h^{t+1}) u(c_{t+1}(h^{t+1})) \pi_{t+1}(h^{t+1})$$

$$= \sum_j \pi_{ij} V(W0 - c_0, \epsilon_j)$$

Stochastic Dynamic programing

- We end up with the following Bellamn equation:

$$V(W_0, \epsilon_0) = \max_c \epsilon_0 u(c) + \beta \sum_j V(W_0 - c_0, \epsilon_j)$$

- where we can also take FOC to get

$$\epsilon_0 u'(c) = \beta \sum_j V_1(W_0 - c_0, \epsilon_j)$$

$$V_1(W_0, \epsilon_0) = \beta \sum_j V_1(W_0 - c_0, \epsilon_j)$$

References